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Orthogonal polynomials and deformed oscillators

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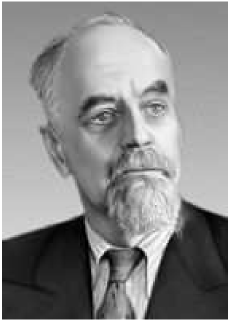
In the period when the authors were students of physical faculty Yu. V. Novozhilov was the head of the Department "Field theory and elementary particle physics".

The great achievement Yu. V. Novozhilov is that the education traditions, which was created by academicians V.I. Smirnov and V.A. Fock, are still supported in the theoretical Department of physical faculty.



High level of physical and mathematical education that was obtained by authors in the years of study, has led to the following facts: the author - physicist (EVD) became interested in the achievements of modern mathematical physics, on the contrary, the other author - mathematician (VVB) became analyze some problems of theoretical physics. As the result, since 2000 the authors have combined their efforts in the study of some mathematical problems of physics.

In the report we are going to discuss some of the authors results connected with the generalization of the concept of a harmonic oscillator, which is one of the basic objects of quantum mechanics.

The development of quantum physics in the end of last century, in particular, the emergence of quantum groups and quantum algebras (L. D. Faddeev school, V.G. Drinfeld [1983/87]), naturally led the various generalizations of the notion of a quantum harmonic oscillator (L.C. Biedenharn, A.J. Macfarlane, P.P. Kulish and E.V. Damaskinsky [1989/90])associated with q-deformations of the canonical commutation relations of the harmonic oscillator.
Note that attempts to generalize the Heisenberg commutation relations have been made also early (G. Iwata (1951); V.V. Kuryshkin (1976); J. Cigler (1979); M. Arik and D. D. Coon (1976)).

Further research led to the construction of various generalizations of Heisenberg algebra, connected with classical orthogonal polynomials of Askey - Wilson scheme and their q-analogues. In the theory of generalized oscillators, these polynomials play the same role as the Hermite polynomials in the theory of usual oscillators.
One of the authors (VVB, 2001) developed a general scheme for the construction of oscillator - like algebras for an arbitrary family of orthogonal polynomials on the real axis (hereafter such algebras is called generalized oscillators).

After that, we applied this approach to
1) some of the classical orthogonal polynomials of a continuous argument (The Laguerre polynomials, Chebyshev (first and second kind), Legendre, Gegenbauer and Jacobi);
2) some of the classical orthogonal polynomials of a discrete argument (Meixner, Charlier and Kravchuk polynomials);
3) some q-analogues of the classical orthogonal polynomials (discrete and continuous q-Hermite polynomials, q-Charlier polynomials);
4) generalized Fibonacci polynomials;
5) two variable Chebyshev - Koornwinder polynomials.

In some cases, for these generalized oscillators we have been constructed and investigated some properties of the corresponding coherent states of Barut - Girardello and Klauder - Gaseau types.

The construction of generalized oscillator, connected with orthogonal polynomial system

Let given system of polynomials $\{P_k\}_{k=0}^{\infty}$, that form an orthogonal basis in the Hilbert space $\mathcal{H}_\mu = \mathrm{L}^2(\mathbb{R}; \mu(dx))$. For a given orthogonality measure of these polynomials (using the method proposed by one of the authors(VVB, 2001)) it is possible to build deformed oscillator-like system for which these polynomials play the same role as the Hermite polynomials for the standard quantum harmonic oscillator.

Let us give the main stages of such constructing.

Now, let μ — probability measure on $\mathbb{R}$ with finite moments $\mu_n = \int_{-\infty}^{\infty} x^n \, d\mu$. These moments define uniquely a positive sequence $\{b_n\}_{n=0}^{\infty}$ and a system of orthogonal polynomials with recurrent relations

$$x \, P_n(x) = b_n \, P_{n+1}(x) + b_{n-1} P_{n-1}(x).$$

The polynomials $P_n(x)$ form an orthogonal basis in the Hilbert space $\mathcal{H}_\mu$.

Then in the Hilbert space $\mathcal{H}_\mu$, one can define creation and destruction operators conjugate to each other $a_\mu^{\pm} = \frac{1}{\sqrt{2}} \left(x_\mu \pm i p_\mu \right)$, the operator N_μ, numbering basis states, and self-adjoint Hamiltonian $H_\mu = x_\mu^2 + p_\mu^2 = a_\mu^+ a_\mu^- + a_\mu^- a_\mu^+$, by the relations

$$a_\mu^+ P_n(x) = \sqrt{2} b_n P_{n+1}(x); \;\; a_\mu^- P_n(x) = \sqrt{2} b_{n-1} P_{n-1}(x);$$

$$N_\mu P_n(x) = n P_n(x), \quad H_\mu P_n(x) = \lambda_n P_n(x),$$

where $\lambda_0 = 2 b_0^2, \quad \lambda_n = 2(b_{n-1}^2 + b_n^2)$.

These operators satisfy the commutation relations of generalized Heisenberg algebra

$$\left[a_\mu^-, a_\mu^+\right] = 2\left(B(N_\mu + I) - B(N_\mu)\right); \quad \left[N_\mu, a_\mu^\pm\right] = \pm a_\mu^\pm,$$

where

$$B(N_\mu)\Psi_n(x) = b_{n-1}^2 \Psi_n(x).$$

The center of this algebra is generated by element $\mathcal{C} = 2B(N_\mu) - a_\mu^+ a_\mu^-$. With appropriate changes, the similar procedure is applicable in the more general case

$$x\, P_n(x) = b_n\, P_{n+1}(x) + a_n\, P_{n+1}(x) + b_{n-1} P_{n-1}(x).$$

Coherent states of Barut - Girardello type for such generalized oscillator are determined by the relations

$$a^-|z>=z|z>, \quad |z>=\mathcal{N}^{-1/2}(|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{(\sqrt{2}b_{n-1})!}P_n,$$

$$\text{where} \qquad \mathcal{N}(|z|^2)=<z|z>=\sum_{n=0}^{\infty}\frac{|z|^{2n}}{(2b_{n-1}^2)!}$$

It is possible to prove that the so-defined coherent states form the overcomplete family of states and are the states that minimize the corresponding the uncertainty relation.

The main difficulty in applying this approach is the solution of the moments problem. This is needed both to construct measures of orthogonality of any two polynomials (not in the classical case) and for determination a measure that is present in the completeness relation for coherent states. There are also some difficulties with representation of the coherent states in terms of hypergeometric or basic-hypergeometric functions.

As an example of a generalized oscillator associated with nonclassical polynomials we consider the case of Fibonacci oscillator, which will defined for orthogonal polynomials connected with generalized Fibonacci numbers.

Fibonacci Numbers

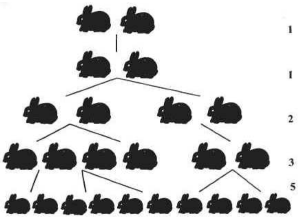
In 1202, Italian merchant and mathematician Leonardo of Pisa (1180-1240), known as Fibonacci, published the essay "Liber Abaca". In this work were collected almost all mathematical information that were known by that time. From here, in particular, European mathematicians used Latin variant of computations, learned about Arabic (or decimal) variant of computation, which significantly simplified arithmetic calculations.

Among the numerous tasks that are listed in this book appears the famous "problem of rabbits", the solution of which determines the sequence of Fibonacci numbers:

$$F_n : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots \; n = 0, 1, 2, \ldots .$$

The elements of this sequence (Fibonacci numbers) are defined by the recurrent relation

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2) \quad F_0 = 1, \; F_1 = 1.$$



It is known that the characteristic equation arising in the solution of this recurrence relation has the form $k^2 = 1 + k$. Using the roots

$$k_1 = \alpha = \frac{1+\sqrt{5}}{2}, \quad k_2 = \beta = \frac{1-\sqrt{5}}{2},$$

of this equation one can represent the Fibonacci numbers in the form

$$F_n = \frac{(\alpha^n - \beta^n)}{\sqrt{5}}.$$

This formula, known as the Binet formula, is notable in that it gives an integer expression (Fibonacci numbers) as a linear combination irrational numbers.

Fibonacci oscillator

Richardson noted that the matrix $\mathfrak{F}_n$ with elements $\frac{1}{F_{i+j+1}}$, where F_n — n-th Fibonacci number, has as its inverse the matrix with integer elements. (Because the same property has Hilbert matrix, he called this matrix the Filbert matrix, creating this term by the rule Fi(bonacci)+(Hi)lbert.) Using this result, Berg showed that the sequence $\frac{1}{F_{n+2}}$ numbers inverse to Fibonacci numbers gives a sequence of moments of discrete probabilistic measure. He also found that this measure is an orthogonality measure for small q-Jacobi polynomials

$$p_n(x; a, b; q) = {}_2\phi_1\left(\frac{q^{-n}, abq^{n+1}}{aq}; q, xq\right),$$

for $a = q,\ b = 1$ и $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$.

Applying the described above method of constructing of the generalized oscillator to the case of polynomials $p_n(x) \equiv p_n(x; a, b; q)$, we get relevant oscillator-like system and a set of their coherent states. We will call this system the **oscillator Fibonacci**..

Let us note that sometimes as Fibonacci oscillator one called two-parameter deformed oscillator ($(p;q)$-oscillator), with basis number $[n]_{q,p} = \dfrac{q^n - p^{-n}}{q - p^{-1}}$. This basis number satisfies the relation

$$[n+1]_{q,p} = (q+p)^{-1}[n]_{q,p} - qp^{-1}[n-1]_{q,p}, \quad [0]_{p,q} = 0, \quad [1]_{p,q} = 1,$$

which is a special case of generalized Fibonacci numbers determined by the relation

$$\mathcal{F}^{a,b}_{n+1} = a\,\mathcal{F}^{a,b}_{n} + b\,\mathcal{F}^{a,b}_{n-1}.$$

In our case, the recurrent relation for the polynomials $p_n(x) \equiv p_n(x; a, b; q)$ have the form

$$-xp_n(x) = A_n p_{n+1}(x) - (A_n + C_n)p_n(x) + C_n p_{n-1}(x), \quad p_0(x) = 1$$

$$A_n = q^n \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n})(1 - abq^{2(n+1)})},$$
$$C_n = aq^n \frac{(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})}.$$

Let us denote $p_n(x) = \gamma_n \Psi_n(x)$, where $\gamma_0 = 1$ и

$$\gamma_n = \sqrt{\frac{C_1 C_2 \cdots C_n}{A_0 A_1 \cdots A_{n-1}}} = \left(a^n q^n \frac{1 - abq}{1 - abq^{2n+1}} \frac{(q, q)_n (bq, q)_n}{(aq, q)_n (abq, q)_n}\right)^{1/2}.$$

Note, that in the case $a = q,\ b = 1$ we have

$$\gamma_n = q^n \frac{(q, q)_n}{(q^2, q)_n} \sqrt{\frac{1 - q^2}{1 - q^{2(n+1)}}}.$$

Then for $\Psi_n(x)$ we obtain

$$x\Psi_n(x) = -b_n\Psi_{n+1}(x) + a_n\Psi_n(x) - b_n\Psi_{n-1}(x),$$

where $\Psi_0(x) = 1$, $a_n = A_n + C_n$ and $b_{n-1} = \sqrt{A_{n-1}C_n}$.
For $a = q,\ b = 1$ one have

$$a_n = \frac{q^n}{1-q^{2(n+1)}}\left(\frac{(1-q^n)^2}{1-q^{2n+1}} + \frac{(1-q^{n+2})^2}{1-q^{2n+3}}\right),$$

$$b_{n-1} = \frac{q^n}{1-q^{2n+1}}\frac{(1-q^n)(1-q^{n+1})}{\sqrt{(1-q^{2n})(1-q^{2(n+1)})}}.$$

Now we define $X_\mu = \mathrm{Re}(\widetilde{X}_\mu - \widetilde{P}_\mu)$, $P_\mu = (-i)\mathrm{Im}(\widetilde{X}_\mu - \widetilde{P}_\mu)$, where operators $\widetilde{X}_\mu$ и $\widetilde{P}_\mu$ are given by

$$\widetilde{X}_\mu \Psi_n = b_{n-1}\Psi_{n-1} + a_n \Psi_n + b_n \Psi_{n+1};$$

$$\widetilde{P}_\mu \Psi_n = i\left(b_{n-1}\Psi_{n-1} + a_n \Psi_n - b_n \Psi_{n+1}\right).$$

As a result, using the above mentioned relations, we define the algebra of Fibonacci oscillator.

Coherent states of Barut - Girardello type for this oscillator are given as

$$a^{-}|z>= z|z>, \quad |z>= \mathcal{N}^{-1/2}(|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{(\sqrt{2}b_{n-1})!}\Psi_n.$$

with normalization constant

$$\mathcal{N}(|z|^2) = {}_6\phi_1\left(\begin{matrix}-q, -q^{3/2}, q^{3/2}, q^{3/2}, -q^{3/2}, -q^2\\ q^2\end{matrix}\,\middle|\, q;\, \frac{|z|^2}{2}\right),$$

where

$${}_6\phi_1\left(\begin{matrix}a_1, a_2, a_3, a_4, a_5, a_6\\ b_1\end{matrix}\,\middle|\, q; z\right) =$$

$$\sum_{k=0}^{\infty}\frac{(a_1, a_2, a_3, a_4, a_5, a_6; q)_k}{(b_1; q)_k}(-1)^{2k}q^{-2k\binom{k}{2}}.$$

Above we have used the standard notation for q-symbol Pochhammer q-symbol:

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n.$$

In the case $a = q,\ b = 1$, the expressions for the coherent states are simplified and have the form

$$|z> = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} q^{-n(n+3)/2}\, p_n(x;q,1|q)\, \frac{(q^3;q)_{2n}}{(q;q)_n^2} \left(\frac{z}{\sqrt{2}}\right)^n,$$

with the same expression for normalization constant.

The dimension of generalized oscillators the algebras

In the recent article

G. Honnouvo, K. Thirulogasanthar, *On the dimensions of the oscillator algebras induced by orthogonal polynomials*, J. Math. Phys. **55** , 093511 (2014)

authors discussed the question: under which conditions the oscillator algebra connected with orthogonal polynomials on real line is finite-dimensional.
For the case of orthogonal polynomials defined by recurrent relations

$$x\Psi_n(x) = b_n\Psi_{n+1}(x) + b_{n-1}\Psi_{n-1}(x); \quad \Psi_0(x) = 1, \quad b_{-1} = 0$$

G. Honnouvo, K. Thirulogasanthar proved the following theorems.

Theorem

Let two variable sequence $\left\{A_n^{(j))}\right\}_{n=0,\,j=1}^{\infty}$ *defined by*

$$A_n^{(0)} = b_n^2 - b_{n-1}^2, \ldots, A_n^{(j)} = A_{n+1}^{(j-1)} - A_n^{(j-1)},$$

$j = 1, 2, \ldots, \quad n = 0, 1, \ldots.$ *If for every fixed* $j > 0$, *the sequence* $\left\{A_n^{(j))}\right\}_{n=0}^{\infty}$ *is not constant, that is* $A_n^{(j)} \neq$*const,* $n = 0, 1, \ldots$, *then the generalized oscillator algebra* $\mathfrak{A}$ *is of infinite dimension.* $\square$

Theorem

The generalized oscillator algebra $\mathfrak{A}$ *is of finite dimension if and only if*

$$b_n^2 = a_0 + a_1 n + a_2 n^2, \qquad a_0, a_1, a_2 \in \mathbb{R}. \tag{1}$$

And in this case the dimension of the algebra $\mathfrak{A}$ *is four.*$\square$

Remark

Unfortunately, the only a necessary part of the above theorem was correct. To was true a sufficient part of the theorem it is need to clarify the condition (1). Namely, the coefficients in (1) must satisfy the following equality

$$a_1 = a_0 + a_2. \tag{2}$$

So correct theorem is

Theorem

The generalized oscillator algebra $\mathfrak{A}$ is of finite dimension if and only if

$$b_n^2 = (a_0 + a_2 n)(1 + n), \qquad a_0, a_2 \in \mathbb{R}. \tag{3}$$

And in this case the dimension of the algebra $\mathfrak{A}$ is four. □

Moreover we prove

Theorem

The above (corrected) theorems are hold for generalized oscillator algebra corresponding to the orthogonal polynomial system $\{\Psi_n(x)\}_{n=0}^{\infty}$, which satisfy nonsymmetric recurrence relations

$$x\Psi_n(x) = b_n\Psi_{n+1}(x) + a_n\Psi_n(x) + b_{n-1}\Psi_{n-1}(x); \quad \Psi_0(x) = 1, \quad b_{-1} = 0.$$

From this theorem, after inspection of recurrent relations of classical orthogonal polynomials we obtain

Corollary

Oscillator algebra associated with a system of orthogonal polynomials depending on one variable is finite-dimensional (namely, four - dimensional) only for the case of Hermite or Laguerre polynomials or for those polynomials, which generate the oscillator algebra that is isomorphic to the above mentioned algebras.

Our talk is based on the following articles

V. V. Borzov, ”Orthogonal polynomials and generalized oscillator algebras".
Integral Transf. and Special Functions, **12**(2), 115-138 (2001).

Е.В. Дамаскинский, В.В. Борзов, Осциллятор Фибоначчи. стр.54-65 в сб."Квантовая теория и космология". Сборник статей, посвященный 70-летию профессора А. А. Гриба. САНКТ-ПЕТЕРБУРГ 2009.

V. V. Borzov, E. V. Damaskinsky, "On dimensions of oscillator algebras".
Proc. of "Days on Difraction 2014" (in press)